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Measuring comodules — their applications

Marjorie Batchelor

Department of Mathematics, King's College, Strand, London WC2R 2LS, UK

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Abstract

Measuring comodules are defined and shown to provide a useful generalization of the set of maps between modules with a broad range of applications. Three applications are described. Connections on bundles are described in terms of measuring comodules, enabling curvature to be defined under general algebraic circumstances. Loop algebras are realized via a short exact sequence of measuring comodules, with the central extension given by the curvature. Finally dual comodules provide a method of dualizing representations, which when applied to representations of loop algebras yield positive energy representations, and when applied to representations of totally disconnected groups leads to the smooth dual. © 2000 Published by Elsevier Science B.V.

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1. Introduction

For some time measuring coalgebras have been employed as sets of generalized maps between algebras [1]. The purpose of this paper is to introduce measuring comodules which provide a set of generalized module maps from a module M over an algebra A to a module N over a different algebra B.

The categorical implication of this construction is presented in [4]. This paper presents a more practical approach not only in the construction of categorical interest, but it also has wide range of potential applications, three of which are described here.

The first application describes connections on bundles. A connection is that construction which is required to describe covariant differentiation of a section of a vector bundle. As such it is amenable to algebraic description, and indeed gives an example of a measuring comodule. The curvature of a connection can likewise be described as an element of a measuring comodule.

The second application generalizes an alternative construction of the universal enveloping algebra of a Lie algebra using measuring coalgebras. That construction proceeds as follows.

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Given a representation of a Lie algebra L as derivations of an algebra A, the universal enveloping algebra UL arises as the subcoalgebra-subalgebra of the universal measuring coalgebra $P(A, A)$ generated by L. In an earlier paper quantum group-like objects were shown to arise by considering subcalgebra-subalgebras of $P(A, A)$ generated by certain sets of difference operators [2]. Here, similar sets of difference operators are used to generate subcomodule-subalgebras of the universal measuring comodule $Q(M, M)$ for a suitable A module M. The resulting algebras are closely related to loop algebras and their central extensions. The cocycle defining the central extension arises as the trace of a curvature.

The last application concerns the dual comodule of an A module M, $Q(M, \mathbf{C})$, where A is an algebra over $\mathbf{C}$. This comodule itself becomes an A module with a strong finiteness property: every element is contained in a finite dimensional A submodule. Two examples are considered. In the first case M is a representation of a totally disconnected group G, e.g., an algebraic group over the p-adic numbers. The construction of interest concerns the dual comodule of M considered as a module for the group algebra $A = \mathbf{C}K$, where K is a compact open subgroup of G. The resulting dual comodule turns out to be not only a representation of K, but of the whole of G. It is closely related to the smooth dual.

In the second example, M is a level k representation of a loop algebra $L[x, x^{-1}]$ (where L is a finite dimensional semi-simple Lie algebra). The dual comodule for M considered as an $A = UL[x]$ module is not only an $L[x]$ module but a level k representation of $L[x, x^{-1}]$. Moreover it contains as a functorially identifiable submodule a positive energy piece. When M is positive energy, this piece is the dual positive energy level k representation of $L[x, x^{-1}]$.

This paper is organised as follows. Section 2 describes measuring coalgebras and comodules. While essentially the constructions are the same as those described in [4], here they are presented in a simplified algebraic context rather than the more general categorical setting. Connections are described in Section 3, the association with loop algebras in Section 4, and finally the two applications of dual comodules in Section 5.

2. Measuring coalgebras and measuring comodules

Although measuring coalgebras (and dual coalgebras in particular) have been around for a long time, I will develop the theory of measuring coalgebras and measuring comodules in parallel, as the first serves as an accessible model for the second.

Definition 2.1 (Measuring coalgebras). If A and B are algebras over a field $\mathbf{k}$, a *measuring coalgebra* is a coalgebra C over $\mathbf{k}$ with comultiplication

$$\Delta : C \to C \otimes C, \qquad \Delta c = \sum_{(c)} c_{(2)} \otimes c_{(1)}, \tag{1}$$

and counit $\epsilon : C \to \mathbf{k}$ together with a linear map, called a *measuring map*

$$f : C \to \mathrm{Hom}_{\mathbf{k}}(A, B) \tag{2}$$

such that

1. $\phi c(aa') = \sum_{(c)} \phi c_{(2)}(a) \phi c_{(1)}(a')$,

2. $\phi c(1_A) = \epsilon(c)1_B$

for a, a' in A, 1_A, 1_B, the appropriate identity elements. The map ϕ is said to measure.

Statements (1) and (2) are equivalent to the statement that the transpose map

$$\phi : A \to \mathrm{Hom}_{\mathbf{k}}(C, B) \tag{3}$$

is an algebra homomorphism, where the multiplication in $\mathrm{Hom}_{\mathbf{k}}(C, B)$ is given by

$$\mu \bullet \nu(c) = \sum_{(c)} \mu(c_{(2)})\nu(c_{(1)}) \tag{4}$$

with identity

$$1(c) = \epsilon(c)1_B. \tag{5}$$

The following proposition summarizes results about measuring coalgebras described in [4].

Proposition 2.2.

1. *Given algebras A, B, there is a category of measuring coalgebras $C(A, B)$ whose objects are measuring coalgebras (C, ϕ) and whose maps $r : (C, \phi) \to (C', \phi')$ are coalgebra maps $r : C \to C'$ such that the following diagram commutes.*

$$\begin{tikzcd} C \arrow[rr] \arrow[dr] & & \mathrm{Hom}_{\mathbf{k}}(A, B) \\ & C' \arrow[ur] & \end{tikzcd}$$

2. *The subcategory of finite dimensional measuring coalgebras is dense in $C(A, B)$. Essentially, every measuring coalgebra is a limit of finite dimensional subcoalgebras (for a discussion of density see* [6, Chapter 5]).
3. *The category $C(A, B)$ has a final object, $(P(A, B), \pi)$ called the universal measuring coalgebra.*

   Thus there is a correspondence of sets

$$\textit{Coalgebra maps}(C, P(A, B)) \leftrightarrow \textit{Algebra maps}(A, \mathrm{Hom}_{\mathbf{k}}(C, B)) \tag{6}$$

4. *If A_i, $i = 1, 2, 3$, are algebras there is a map*

$$m : C(A_2, A_3) \times C(A_1, A_2) \to C(A_1, A_3)$$

   In particular, $P(A, A)$ is a bialgebra.
5. *The universal measuring coalgebra $P(A, A)$ is a bialgebra.*

Proof. Full proofs can be found in [4]. However, as the presentation in [4] is highly categorical and more general than is necessary here, direct proofs of (2) and (3) are indicated below.

(2) To establish density it is sufficient to show that every element c of a coalgebra C is contained in a finite dimensional subcoalgebra C_1. As this result is the essential property

of coalgebra, the proof from Sweedler [9, p. 46] is repeated here. Let $C' = \mathrm{Hom}_{\mathbf{k}}(C, k)$ be the dual algebra and consider the action of C' on C given by

$$c' \cdot c = \sum_{(c)} c'(c_{(2)})c_{(1)}. \tag{7}$$

Evidently the C' module V generated by c is finite dimensional, and

$$C' \to \mathrm{End}(V) \tag{8}$$

is an algebra homomorphism of cofinite dimensional kernel J.

Let $J^{\perp}$ be the subspace of C on which J is identically zero. Finally notice that $J^{\perp} \to \mathrm{Hom}(C'/J, \mathbf{k})$ is finite dimensional and c is in $J^{\perp}$. A subcoalgebra of a measuring coalgebra is itself a measuring coalgebra (with the restriction of the measuring map), hence the result.

(3) This depends on two categorical properties of coalgebras.

(a) Arbitrary coproducts exist in the category of coalgebras and coalgebra maps.

(b) Coequalizers also exist in this category.

The construction of $P(A, B)$ proceeds as follows. Consider the collection $\{(C_\lambda, \phi_\lambda)\}$ of finite dimensional measuring coalgebras. Form the coproduct $\sqcup_\lambda C_\lambda$, this is a measuring coalgebra. Now consider the set $\{\rho(\lambda, \mu)\}$ of maps $\rho(\lambda, \mu) : C_\lambda \to C_\mu$ of finite dimensional measuring coalgebras. Form the coproduct $\sqcup_{\rho(\lambda,\mu)} C_\lambda$, this is also a measuring coalgebra. There are two maps

$$\alpha, \beta : \sqcup_{\rho(\lambda,\mu)} C_\lambda \to \sqcup_\lambda C_\lambda \tag{9}$$

On C_λ, α is just the inclusion $C_\lambda \to \sqcup_\lambda C_\lambda$, while β is the composition of $\rho(\lambda, \mu)$ with the inclusion $C_\mu \to \sqcup_\lambda C_\lambda$.

The claim is that the coequalizer $P(A, B)$ has the desired universal property. If (D, ψ) is a measuring coalgebra, then D is the union of finite dimensional subcoalgebras D_ν. Evidently there is a map $r_\nu : D_\nu \to P(A, B)$ and this map is unique. The uniqueness of r_ν guarantees the map $\rho : D \to P(A, B)$ given by $\rho(d) = r_\nu(d)$ if d in D_ν is well defined. □

Example 2.3.

1. $P(A, B)$ is intended to generalize the set of algebra homomorphisms from A to B, and so it does. Let $C_0 = \mathbf{k}g$ be the one-dimensional coalgebra with $\Delta g = g \otimes g$, $\epsilon(g) = 1$. Then a map $\phi : C_0 \to \mathrm{Hom}_{\mathbf{k}}(A, B)$ measures if and only if $\phi(g)$ is an algebra homomorphism. Thus $P(A, B)$ contains all algebra homomorphisms.
2. Let $C_1 = \mathbf{k}g \otimes \mathbf{k}\gamma$, g as above, and let $\Delta\gamma = g \otimes \gamma + \gamma \otimes g$, $\epsilon(\gamma) = 0$. Then $\phi : C_1 \to \mathrm{Hom}_{\mathbf{k}}(A, B)$ measures if and only if $\phi(g)$ is an algebra homomorphism and $\phi(\gamma)$ is a derivation with respect to $\phi(g)$. That is,
$$\phi(\gamma)(aa') = \phi(\gamma)(a)\phi(g)(a') + \phi(g)(a)\phi(\gamma)(a'). \tag{10}$$
3. More generally if L is a Lie algebra over $\mathbf{k}$, then $L \oplus C_0$ can be given the structure of coalgebra with comultiplication $\Delta\gamma = \gamma \otimes g + g \otimes \gamma$ and $\epsilon(\gamma) = 0$ for γ in L. Suppose
$$\phi : L \oplus C_0 \to \mathrm{Hom}_{\mathbf{k}}(A, A) \tag{11}$$

is a measuring map such that

$$\phi[\nu, \gamma] = [\phi\nu, \phi\gamma], \qquad \phi(g) = Id \tag{12}$$

for ν, γ in L. By the universal property of $P(A, A)$, there is a map of measuring coalgebras

$$\rho : L \oplus C_0 \to P(A, A) \tag{13}$$

However, $P(A, A)$ is an algebra, and in fact the following is true.

Proposition 2.4. *If the map ϕ is injective on L, the subalgebra of $P(A, A)$ generated by the image of ρ is isomorphic to the universal enveloping algebra UL.*

The proof follows from the universal property of $P(A, A)$ and facts about bialgebras [2]. In [2] I considered subalgebras of $P(A, A)$ generated by measuring coalgebras $L \oplus \mathbf{C}K$, where K is a group, $\mathbf{C}K$ has the usual comultiplication $\Delta k = k \otimes k$ for k in K, and elements of L have a slightly skew version of the usual comultiplication for derivations

$$\Delta E = E \otimes k + k^{-1} \otimes E, \tag{14}$$

which is characteristic of difference operators. These objects resemble quantum groups. The construction in Section 4 of this paper uses the same procedure to construct subalgebras of the universal measuring comodule (defined below), which are related to central extensions of loop algebras.

Definition 2.5 (Measuring comodules). Let M be an A module and let N be a B module (all modules and algebras are vector spaces over $\mathbf{k}$). When it is necessary to emphasize the algebra over which M and N are modules, write ${}^AM$, ${}^BN$. Let (C, ϕ) be a measuring coalgebra in $C(A, B)$. Recall that a *comodule* over C is a vector space with a comultiplication

$$\Delta : D \to C \otimes D, \qquad \Delta(d) = \sum_{(d)} d_{(1)} \otimes d_{(0)}. \tag{15}$$

In addition, I will assume that $(\epsilon \otimes 1)\Delta = 1$. When it is necessary to keep the track of the coalgebra over which D is a comodule, we write ${}_CD$. A $\mathbf{k}$-linear map

$$\psi : D \to \mathrm{Hom}_{\mathbf{k}}(M, N) \tag{16}$$

measures if

$$\psi(am) = \sum_{(d)} \phi d_{(1)}(a)\psi d_{(0)}(m). \tag{17}$$

The pair (D, ψ) is called a measuring comodule, and ψ is called a measuring map.

Equivalently ψ measures if and only if the corresponding transpose map

$$\psi : M \to \mathrm{Hom}_{\mathbf{k}}(D, N) \tag{18}$$

is a map of A modules, where the A module structure on $\mathrm{Hom}_{\mathbf{k}}(D, N)$ is given by

$$a \bullet \beta(d) = \sum_{(d)} \phi d_{(1)}(a)\beta d_{(0)}. \tag{19}$$

Again, the results from Hyland and Batchelor [4] are summarized in the following proposition.

Proposition 2.6.

1. *Given a measuring coalgebra C in* $C(A, B)$ *there is a category* ${}_C D(M, N)$*, whose objects are measuring comodules* (D, ψ) *and whose maps* $\sigma : (D, \psi) \to (D', \psi')$ *are comodule maps* $\sigma : D \to D'$ *such that the following diagram commutes.*

$$\begin{tikzcd} D \arrow[rr] \arrow[dr] & & \mathrm{Hom}_k(M, N) \\ & D' \arrow[ur] & \end{tikzcd}$$

2. *The subcategory of* ${}_C D(M, N)$ *whose objects are the finite dimensional measuring comodules is a dense subcategory of* ${}_C D(M, N)$.
3. *The category* ${}_C D(M, N)$ *has a final object,* ${}_C Q(M, N)$*. This has the property that there is a correspondence*

$$C--comodule\ \ maps(D, {}_C Q(M, N)) \leftrightarrow A--module\ \ maps(M, \mathrm{Hom}(D, N)) \tag{20}$$

4. *If* M_i *are modules over algebras* $A_i, i = 1, 2, 3$, *and if* C, C' *are in* $C(A_1, A_2)$, $C(A_2, A_3)$, *respectively, then there is a map*

$${}_C D(M_2, M_3) \times_{C'} D(M_1, M_2) \overset{m}{\to}_{m(C \times C')} D(M_1, M_3) \tag{21}$$

In particular, ${}_C Q(M, M)$ *is a comodule algebra for* $C \to \mathrm{Hom}(A, A)$ *a measuring coalgebra, M is an A module.*

5. *If* $A = B$ *and* $M = N$*, then* ${}_C Q(M, N)$ *is a comodule algebra.*

Proof. (2) Again the important step is to show that if D is a C comodule, then each d in D is contained in a finite dimensional subcomodule.

Define an action $\bullet$ of C', the linear dual of C on D via

$$a \bullet d = \sum_{(d)} a(d_{(1)}) d_{(0)}. \tag{22}$$

Choose an element d of d_0, and let $D_0 = C' \bullet d_0$. Evidently D_0 is a finite dimensional C' module and $d_0 = 1 \bullet d_0$ is in D_0.

The full linear dual D' (of D) is also a C' module with the action given explicitly by

$$a * d(d) = \sum_{(d)} a(d_{(1)}) d(d_{(0)}). \tag{23}$$

The subset $D_0^{\perp} = \{d \in D' : d(D_0) = 0\}$ is a submodule, and $D_0' = D'/D_0^{\perp}$. But $(D_0^{\perp})^{\perp}$ is then a subcomodule of D, and D is contained in $(D_0^{\perp})^{\perp}$. Since $(D_0^{\perp})^{\perp}$ includes in $(D'/D_0^{\perp})'$, $(D_0^{\perp})^{\perp}$ must be a finite dimensional comodule as required. The proofs of (1) and (3) are identical in format to the corresponding statements for measuring coalgebras. $\square$

Example 2.7.

1. Let $C_0 = \mathbf{k}g$ as in Example 2.3(1) and suppose that $\phi : C \to \mathrm{Hom}_{\mathbf{k}}(A, B)$ measures, so that $\phi(g)$ is an algebra homomorphism. Let D be the comodule with $D = \mathbf{k}d$ and comultiplication $\Delta d = g \otimes d$. Let $\psi : D \to \mathrm{Hom}_{\mathbf{k}}(M, N)$ be a linear map. Recall that the pullback of N, $\phi(g)^*N$ is an A module. Then ψ measures if and only if
$$\psi(d) : M \to \phi(g)^*N \tag{24}$$
is a map of A modules.
2. If $A = B$ and if C contains the measuring comodule C_0 with $\phi(g) = 1$, the measuring comodule ${}_CQ(M, N)$ contains the vector space H of all genuine A module maps from M to N as follows.

   Any vector space, e.g. H, is trivially a C comodule with comultiplication $\Delta h = g \otimes h$. The inclusion $\psi : H \to \mathrm{Hom}_{\mathbf{k}}(M, N)$ is then a measuring map. By the universal property there is a unique map of measuring comodules $\rho : H \to_C Q(M, N)$. Since ψ is an inclusion, so must ρ be.
3. Any algebra can be considered as a module over itself acting by left multiplication. If $C \to \mathrm{Hom}(A, B)$ is a measuring coalgebra, by considering C as a comodule over itself, $C \to \mathrm{Hom}(A, B)$ is also a measuring comodule.
4. For an element a of an algebra A, let ι_a denote the inner derivation
$$\iota_a(b) = [a, b] = ab - ba. \tag{25}$$
Let I_A denote the Lie algebra of inner derivations of A. As in Example 2.3(3), let C be the measuring coalgebra $C = I_A \oplus C_0$, $C \to \mathrm{Hom}_{\mathbf{k}}(A, A)$. Now put a C comodule structure on A,
$$\Delta(a) = g \otimes a + \iota_a \otimes 1. \tag{26}$$
Now let M be an A module. With the comodule structure above, the inclusion $A \to \mathrm{Hom}_{\mathbf{k}}(M, M)$ sending a to left multiplication by a gives A, the structure of a measuring comodule. This construction generalizes the observation that for modules over commutative rings, left multiplication is a module map.

Remark 2.8.

1. *If $\tau : (C, f) \to (C', f')$ is a map of measuring coalgebras, in particular, τ is a comodule map, so that ${}_CQ(M, N)$ can be considered as a C' comodule. Since τ is a map of measuring coalgebras ${}_CQ(M, N)$ is in fact in ${}_{C'}D(M, N)$, and hence by the universal property there is a unique map ${}_CQ(M, N) \to_{C'} Q(M, N)$. All universal measuring comodules ${}_CQ(M, N)$ thus map to ${}_{P(A,B)}Q(M, N)$, which will often be denoted as $Q(M, N)$.*
2. *The construction $Q(M, N)$ serves as the set of "module maps from an A module M to a B module N" even when A is not the same as B. The paper* [4] *arose from the desire to put this curiosity into a sound categorical context.*

3. Connections

Given an algebra A of functions, and a set V of derivations of A (vector fields), a connection is that which is needed to define covariant differentiation by elements of V on a module M (e.g., sections of a bundle) over A. This is a completely algebraic statement and as such lends itself to restatement in terms of measuring comodules.

Definition 3.1 (Loose connections). Let A be an algebra and let M be a module over A. Let C be a measuring coalgebra, and let D be a comodule over C which is also an A module. A loose connection is a measuring map

$$\nabla : D \to \mathrm{Hom}_{\mathbf{k}}(M, M), \tag{27}$$

which additionally satisfies the requirement that ∇ be a map of A modules in the sense that

$$\nabla(a\xi)(m) = a\nabla\xi(m). \tag{28}$$

Example 3.2.

1. *Connections on a vector bundle*. Let $A = C^\infty(Y)$, where Y is a smooth manifold and let V be the Lie algebra of vector fields on Y and let $C = V \oplus \mathbf{C}1$. Let $D = V \oplus A$ with the comultiplication

$$\begin{aligned} &\Delta : D \to C \otimes D, \qquad \Delta(\psi) = 1 \otimes \psi + \psi \otimes 1, \quad \psi \in V, \\ &\Delta(a) = 1 \otimes a + \iota_a \otimes 1, \quad a \in A. \end{aligned} \tag{29}$$

   Notice that D is an A module. Let E be a vector bundle over Y and let $\Gamma(Y, E)$ denote the smooth sections of E over Y. Thus $M = \Gamma(Y, E)$ is a module for A. In this setting, loose connections are precisely Koszul connections (see [8]).
2. *Connections on a principle bundle* (*see* [7]). Let Y be a manifold and let P be a principle G bundle over Y. Let $M = C^\infty(P)$. Observe that $C^\infty(Y)$ includes in M as those functions which are constant on the fibres of P, hence M is a $C^\infty(Y)$ module. In addition, the group algebra $\mathbf{R}G$ acts on M via right translation. The action of $\mathbf{R}G$ commutes with the action of $C^\infty(Y)$.

   Let $A = C^\infty(Y) \otimes \mathbf{R}G$. Let V be the Lie algebra of vector fields on Y. Observe that the coalgebra C above becomes a measuring coalgebra with measuring map

$$\phi : C \to \mathrm{Hom}(C^\infty(Y) \otimes \mathbf{R}G, C^\infty(Y) \otimes \mathbf{R}G), \qquad \phi(\psi) f \otimes g = \psi f \otimes g. \tag{30}$$

   Let D be the comodule of Example 3.2(1). This is a $C^\infty(Y) \otimes \mathbf{R}G$ module with the trivial action of G on $C^\infty(Y)$ and V. Also notice that D contains C as a subcomodule (with its usual coproduct). A loose connection

$$\nabla : D \to \mathrm{Hom}(C^\infty(P), C^\infty(P)) \tag{31}$$

   in this setting corresponds to a connection on the principle bundle if and only if additionally ∇ restricted to the subspace C defines a measuring coalgebra

$$\nabla : C \to \mathrm{Hom}(C^\infty(P), C^\infty(P)) \tag{32}$$

3.1. Curvature

Curvature can be defined for any measuring comodule equipped with Lie bracket, in particular for loose connections. Recall that any measuring comodule, D in particular, comes with a map of measuring comodules

$$\rho : D \to Q(M, M) \tag{33}$$

Recall that $Q(M, M)$ is a comodule algebra. If D contains a subspace V on which a Lie bracket is given for ξ, ψ in V, we write

$$\Omega(\xi, \psi) = \rho(\xi)\rho(\psi) - \rho(\psi)\rho(\xi) - \rho([\xi, \psi]) \tag{34}$$

This map

$$\Omega : V \otimes V \to Q(M, M) \tag{35}$$

is the *curvature* of the loose connection ∇ on V.

Remark 3.3. *In all classical cases, the coalgebra C is always $V \oplus \mathbf{C}1$, the comodule D is always $V \oplus A$, where V is the Lie algebra of derivations of A, and one is only interested in the restriction of ∇ to V. There is no harm, however, in allowing this greater generality. In the next section a very different example demonstrates the advantages of being broad minded.*

This section concludes with a result which is well known for conventional connections.

Proposition 3.4. *If V is a set of primitive elements, i.e., with comultiplication*

$$\Delta x = \xi \otimes 1 + 1 \otimes \xi, \tag{36}$$

then $\Omega(\xi, \psi)$ determines a module map

$$\Omega(\xi, \psi) : M \to M \tag{37}$$

Proof. Direct calculation (observing that $Q(M, M)$ is a comodule algebra, i.e., that multiplication preserves the comodule structure) shows that

$$\Delta\Omega(\xi, \psi) = 1 \otimes \Omega(\xi, \psi). \tag{38}$$

But this is exactly the statement that $\Omega(\xi, \psi)$ is a module map. □

4. Generalization of universal enveloping algebras using measuring comodules

In Proposition 2.4, the universal enveloping algebra is constructed as a subalgebra of the universal measuring coalgebra generated by a Lie algebra of derivations. The original Lie algebra can be identified as the subspace of primitive elements.

This construction can be generalized, replacing primitive elements (derivations) with elements E of a measuring coalgebra with the asymmetric comultiplication $\Delta E = E \otimes$

$K + K^{-1} \otimes E$, where K is an (invertible) group-like element. The algebras generated by such E and K resemble quantum groups, and were the subject of [2].

This construction is now generalized again, replacing the universal measuring coalgebra with the universal measuring comodule. The resulting algebra has, as the analogue of its Lie algebra of primitive elements, a Lie algebra related to the central extensions of loop algebras. The central term arises as the trace of the curvature.

Construction 4.1. Let A be an algebra and let M be an A module. Let P_0 be a subcoalgebra-subalgebra of $P(A, A)$, and define

$$\begin{aligned} V_0 &= \{v \in Q(M, M) : \Delta v \in P_0 \otimes Q(M, M)\}, \\ V &= \{v \in Q(M, M) : \Delta v \in P(A, A) \otimes 1 + 1 \otimes Q(M, M) + P_0 \otimes Q(M, M)\}. \end{aligned} \tag{39}$$

Evidently A is contained in V. The subcomodule V is the generalization of primitive elements referred above. Suppose a "trace"

$$\tau : V_0 \to C \tag{40}$$

is given. Let K be the kernel of τ. Define

$$V_{0\tau} = \{v \in V_0 : [K, v] \leq K\}, \qquad V_\tau = \{v \in V : [V_{0\tau}, v] \leq K\}. \tag{41}$$

Observe that V_τ is not an algebra, but the Jacobi identity guarantees that it is closed under Lie bracket. V_0 is a subalgebra of $Q(M, M)$ and hence $V_{0\tau}$ is a Lie subalgebra. It is not hard to check that there is a short exact sequence of Lie algebras

$$0 \to \frac{V_{0\tau}}{K} \to \frac{V_\tau}{K} \to \frac{V_\tau}{V_{0\tau}} \to 0. \tag{42}$$

Moreover, V_τ/K is a central extension of $V_\tau/V_{0\tau}$.

Suppose now $\mu : V_\tau/V_{0\tau} \to V_\tau$ is any linear section of the projection $V_\tau \to V_\tau/V_{0\tau}$. The image $\mu(V_\tau/V_{0\tau})$ inherits a Lie bracket from $V_\tau/V_{0\tau}$: hence the associated curvature Ω_μ takes values in $V_{0\tau}$. While the Ω_μ may depend on the section μ, the trace of the curvature does not. In fact V_τ/K is the central extension of $V_\tau/V_{0\tau}$ with cocycle c defined by

$$c(v, w) = \tau(\Omega_\mu(\mu v, \mu w)) \tag{43}$$

for v, w in V_τ/V_0. Familiar examples arise from looking at particular subspaces of V.

Example 4.2.

1. Let $A = M = \mathbf{C}[x]$. Let

$$P_0 = \{p \in P(\mathbf{C}[x], C) : p(x^n) = 0 \text{ for almost all } n\}. \tag{44}$$

Explicitly P_0 has basis $\{\beta_j\}$ with comultiplication and measuring map given by

$$\Delta\beta_j = \sum_{k=0} \beta_k \otimes \beta_{j-k}, \qquad \phi(\beta_j) = j! \frac{\mathrm{d}}{\mathrm{d}x}\bigg|_0. \tag{45}$$

Define a trace τ on V_0,

$$\tau(v)=\sum_{j=0}^{\infty}\beta_j(v(x^j)). \tag{46}$$

To see this well defined, observe that for v in V_0

$$v(x^n)=\sum_{(v)}v_{(1)}(x^n)v_{(0)}(1). \tag{47}$$

Thus $\beta_j(v(x^n))=0$ for greater than the greatest degree of the $v_{(0)}(1)$, and the sum defining τ is always a finite sum.

This example contains two very well-known examples as Lie subalgebras. First consider $\mathbf{C}[\alpha^{-1}]$. This can be given the structure of a coalgebra with

$$\Delta(\alpha^{-i})=\alpha^{-i}\otimes 1+\sum_{k=0}^{i-1}\beta_k\otimes\alpha^{-i+k},\quad i>0,\qquad \epsilon(\alpha^i)=\delta_{i,0}. \tag{48}$$

Define a map $\phi:\mathbf{C}[\alpha^{-1}]\to\mathrm{Hom}(\mathbf{C}[x],\mathbf{C}[x])$ via

$$\phi(\alpha^i)x^n=\begin{cases}x^{i+n}, & i+n\geq 0,\\ 0 & \text{otherwise.}\end{cases} \tag{49}$$

It is routine, if surprising, to verify that this map measures. Now $\mathbf{C}[\alpha,\alpha^{-1}]$ can be considered as a comodule over $\mathbf{C}[\alpha^{-1}]\oplus P_0$ with comultiplication given by

$$\Delta\alpha^i=1\otimes\alpha^i \tag{50}$$

for $i>0$, otherwise as above. Clearly the measuring map ϕ above extends to all of $\mathbf{C}[\alpha,\alpha^{-1}]$. It is not hard to check that the image of $\mathbf{C}[\alpha,\alpha^{-1}]$ lies in V_τ. The image of $\mathbf{C}[\alpha,\alpha^{-1}]$ in V_τ/K is the familiar central extension of the abelian Lie algebra $\mathbf{C}[\alpha,\alpha^{-1}]$ with cocycle

$$c(\alpha^k,\alpha^j)=k\delta_{k,-j}. \tag{51}$$

2. With P_0 and τ as before, let T be the vector space with basis $\{T_i,i\in\mathbf{Z}\}$, and put a comodule structure on T via

$$\Delta(T_i)=T_i\otimes 1+1\otimes T_i+\sum_{i+k<0}k\beta_k\otimes\alpha^{k+i}+\beta^k\otimes T_{k+i}. \tag{52}$$

Observe that $T\oplus P_0\oplus\mathbf{C}[\alpha^{-1}]$ is in fact a coalgebra if the counit on T is defined to be identically zero. Extending ϕ of the previous example via

$$\phi(T_i)(x^n)=\begin{cases}x^{i+n}, & i+n\geq 0,\\ 0 & \text{otherwise}\end{cases} \tag{53}$$

gives $T\oplus P_0\oplus\mathbf{C}[\alpha^{-1}]$, the structure of a measuring coalgebra, and hence a measuring comodule. Again the image lies in V_τ. The image of T in $V_\tau/V_{0\tau}$ is isomorphic to the

Lie algebra of derivations of $\mathbf{C}[x, x^{-1}]$, and its image in V_τ/K is the variant of the Virasoro algebra with cocycle

$$c(T_m, T_n) = \tfrac{1}{6}(m^3 - m)\delta_{m,-n}. \tag{54}$$

3. Return now to a general algebra A, and suppose that $M = A$, and suppose also that τ is given so that K, V_τ and $V_{0\tau}$ are as described in Construction 4.1. Let L be a finite dimensional (semi-simple) Lie algebra, which is faithfully represented by $\rho : L \to \mathrm{End}(W)$. Let $M(W) = A \otimes W$. Then the identification of $\mathrm{Hom}(M(W), M(W))$ with $\mathrm{Hom}(A, A) \otimes \mathrm{Hom}(W, W)$ provides a map

$$\phi \otimes \rho : V \otimes L \to \mathrm{Hom}(M(W), M(W)) \tag{55}$$

which measures. Moreover, $V \otimes L$, is closed under Lie bracket, as is $V_0 \otimes L$.

If κ is the Killing form on L then $\tau \otimes \kappa$ is well defined on $V_0 \otimes L$, and $\tau \otimes \kappa([V_\tau \otimes L, V_{0\tau} \otimes L]) = 0$. Let $K(L)$ be the kernel of $\tau \otimes \kappa$. There is then a short exact sequence of Lie algebras

$$0 \to \frac{V_{0\tau} \otimes L}{K(L)} \to \frac{V_\tau \otimes L}{K(L)} \to \frac{V_\tau \otimes L}{V_{0\tau} \otimes L} \to 0 \tag{56}$$

The Lie algebra $V_\tau \otimes L/K(L)$ is the central extension of the loop algebra $L \otimes \mathbf{C}[x, x^{-1}] \approx V_\tau \otimes L/V_{0\tau} \otimes L$. It turns out that the cocycle c of the central extension is given by

$$c(v \otimes \xi, w \otimes \psi) = \tau\Omega(\mu v, \mu w)\kappa(\xi, \psi). \tag{57}$$

In case (1), $V_\tau \otimes L/V_{0\tau} \otimes L = L[x, x^{-1}]$, the loop algebra of L, and c is the expected central extension

$$c[x^m \xi, x^n \psi] = m\delta_{-m,n}\kappa(\xi, \psi). \tag{58}$$

5. Dual comodules, positive energy representations, and smooth representations

5.1. Dual coalgebras and dual comodules

If A is an algebra, and M is an A module, then the constructions $P(A, \mathbf{C})$, $Q(M, \mathbf{C})$ have alternative descriptions which make the calculations easy.

Proposition 5.1.
1. $P(A, \mathbf{C}) =: A^* = \{\alpha : A \to \mathbf{C} : \ker \alpha \geq I,\ I \text{ an ideal},\ \dim(A/I) < \infty\}$.
2. $Q(M, \mathbf{C}) =: M^* = \{\mu : M \to \mathbf{C} : \ker \mu \geq W,\ AW \leq W,\ \dim(M/W) < \infty\}$.

Proof. (1) Observe that since A/I is finite dimensional, multiplication in A/I gives the linear dual $(A/I)'$ the structure of a coalgebra with the obvious measuring map into $\mathrm{Hom}(A, \mathbf{C})$. Then, since $(A/I)'$ maps to $P(A, \mathbf{C})$ by the universal property, $A^* = \lim(A/I)' \leq \mathrm{Hom}(A, \mathbf{C})$ maps to $P(A, \mathbf{C})$. But now observe that the measuring map $\pi : P(A, \mathbf{C}) \to$

$\mathrm{Hom}(A, \mathbf{C})$ has its image in A^*. To see this, consider c in $P(A, \mathbf{C})$. Let C be a finite dimensional subcoalgebra of $P(A, \mathbf{C})$ containing c. Then the restriction of the measuring map $\pi : C \to \mathrm{Hom}(A, \mathbf{C})$ corresponds to an algebra homomorphism $\pi : A \to \mathrm{Hom}(C, \mathbf{C})$, the last one being a finite dimensional algebra. Let J be the kernel of π. Since $\pi : A \to \mathrm{Hom}(C, \mathbf{C})$ factors through A/J, $\pi(c) : A \to \mathbf{C}$ must factor through A/J, and $\pi(c)$ is in A^* as required.

(2) The argument is exactly parallel to that of (1). □

Remark 5.2.

1. *Evidently M^* becomes a module for the opposite algebra A^{op} under the action*

$$am = \sum_{(m)} m_{(1)}(a)m_{(0)}. \tag{59}$$

One can ask what representations arise as dual comodules. It is evident that such a representation V must have the property that every element of V lines in some finite dimensional submodule of V. Representations which have this property will be called locally finite.

2. *More generally, given modules M, N over A, B, respectively, $Q(M, N)$ can be considered an A^{op} module*.

The ingredients for the applications of interest are an algebra A and a representation of A on a vector space V and a distinguished subalgebra B. Considered as an A module, $V^* = ({}^A V)^*$ is not very interesting, and may in fact be zero. However, considered as a B module, $({}^B V)^*$ is not only a B module, but also an A module. The property of the subalgebra B which gives $({}^B V)^*$ the structure of an A module is as follows.

Definition 5.3. Say $B \leq A$ is *quasi-normal* if and only if for every a in A, there exists $a_1, \dots, a_n$ such that

$$BaB = \sum_1^l Ba_i = \sum_1^l a_i B. \tag{60}$$

Lemma 5.4. *Suppose B is quasi-normal in A, and let $s : A \to A$ be an anti-automorphism. Then if M is a representation of A, $Q({}^B M, \mathbf{C})$ is an A module*.

Proof. Notice that the action of A^{op} on $\mathrm{Hom}(M, \mathbf{C})$ given by

$$a\mu(m) = \mu(am) \tag{61}$$

coincides with the action of B^{op} on $({}^B M)^*$, whenever μ is in $({}^B M)^*$ and a is in B. Prefacing this action with the anti-automorphism s,

$$a \bullet \mu(m) = \mu(s(a)m) \tag{62}$$

defines an action of A on $\mathrm{Hom}(M, \mathbf{C})$. The claim is that $({}^B M)^*$ is fixed by this action.

Let α be in $({}^B M)^*$ and let a be in A. By Remark 5.2, $\alpha : M \to \mathbf{C}$ vanishes on N, a B submodule with M/N finite dimensional. The problem is to show that there is a B submodule N_a such that M/N_a is finite dimensional and $(a \bullet \alpha)(N_a) = 0$.

Since B is quasi-normal, we write

$$Bs(a)B = \sum_1^l a_i B = \sum_1^l Ba_i. \tag{63}$$

Then, define linear maps

$$\alpha_i : N \to M \to \frac{M}{N}, \qquad \alpha_i(n) = a_i n + N, \tag{64}$$

and let

$$N_i = \ker \alpha_i, \qquad N_a = \cap N_i. \tag{65}$$

Now observe that N_a is a B submodule of M: consider $a_i bn$ for b in B, n in N_a. We can write

$$a_i b = \sum_1^l b_j a_j \tag{66}$$

so that

$$\alpha_i(bn) = a_i bn + N = \sum_1^l b_j a_j n + N. \tag{67}$$

Since n is in N_a, $a_j n$ is in N for all j, and hence so is $b_j a_j n$. Thus bn is in the kernel of a_i for all i. Finally check that N_a is contained in $\ker a \bullet \alpha$. For n in N_a, $a \bullet \alpha(n) = \alpha(s(a)n)$. But $s(a)$ is in $Bs(a)B$, so $s(a) = \sum_1^l b_j a_j$ and $s(a)n = \sum_1^l b_j a_j n$. Since n is in $\ker a_i$ for each i, $a_i n$ is in N for each i, hence $s(a)n$ is in N and $\alpha(s(a)n) = 0$ as required. □

5.2. *Application to totally disconnected groups*

Let G be a totally disconnected group (see [3] for a survey of the representation theory of these objects) with a given compact open subgroup K, and let M be a complex representation of G, hence a representation of $\mathbf{C}G (= A)$ and $\mathbf{C}K (= B)$.

Lemma 5.5. $\mathbf{C}K$ *is quasi-normal in* $\mathbf{C}G$.

Proof. Let g be in G. The double coset KgK is a finite union of either right or left cosets of K and the left coset representatives $\{g_i\}$ may be chosen to be the same as the right coset representatives. Then

$$\mathbf{C}Kg\mathbf{C}K = \sum_1^n \mathbf{C}Kg_j = \sum_1^n g_j \mathbf{C}K \tag{68}$$

as required. □

Corollary 5.6. *$Q({}^{\mathbf{C}K}M, \mathbf{C})$ is a representation of $\mathbf{C}G$ which is locally finite as a representation of $\mathbf{C}K$.*

Proof. All that is needed to meet the conditions of Lemma 5.4 is the choice of an appropriate anti-automorphism. Clearly the map $s(g) = g^{-1}$ is a suitable choice.

The representation $Q({}^{\mathbf{C}K}M, \mathbf{C})$ is almost but not quite the smooth dual of M. The relationship can be described in coalgebraic terms. □

Definition 5.7. If F is a subcoalgebra of C, and D is a C comodule, define the restriction of D to F to be

$$ {}_F|D = \{d \in D : \Delta d \in F \otimes D\}. \tag{69}$$

Thus ${}_F|D$ is a C subcomodule and an F comodule.

In particular, the coalgebra $P(\mathbf{C}K, \mathbf{C}) = (\mathbf{C}K)^*$ contains as an important subcoalgebra the vector space with basis $K^\wedge$, the set of group homomorphisms $\rho : K \to \mathbf{C}$. The trivial homomorphism $\tau : K \to \mathbf{C}$ in particular is in $K^\wedge$. Consider the subcomodule

$$ {}_{\mathbf{C}\tau}|({}^{\mathbf{C}K}M). \tag{70}$$

Proposition 5.8.

1. *If K' is another compact open subgroup of G, then $({}^{\mathbf{C}K}M)^* = ({}^{\mathbf{C}K'}M)^*$.*
2. *If $K' \leq K$, and if τ, τ' are the corresponding trivial homomorphisms, then*

$$ {}_{\mathbf{C}\tau}|({}^{\mathbf{C}K}M)^* \leq {}_{\mathbf{C}\tau'}|({}^{\mathbf{C}K'}M)^*. \tag{71}$$

3. *The union $\cup_{\mathbf{C}\tau}|({}^{\mathbf{C}K}M)^*$ over all compact open K is the smooth dual of M.*

Proof. The only statement which is not immediate is the first. Suppose that $K' \leq K$, then the inclusion induces a map $(\mathbf{C}K)^* \to (\mathbf{C}K')^*$, and any $(\mathbf{C}K)^*$ comodule is automatically a $(\mathbf{C}K')^*$ comodule. Moreover, any $(\mathbf{C}K)^*$ comodule D equipped with a measuring map $\rho : D \to \mathrm{Hom}(M, C)$ is also a measuring comodule for $(\mathbf{C}K')^*$. Thus $({}^{\mathbf{C}K}M)^* \to (\mathbf{C}K'M)^*$.

Less obviously $(\mathbf{C}K'M)^* \to ({}^{\mathbf{C}K}M)^*$. Let $\alpha : M \to C$ vanish on N' which is a $\mathbf{C}K'$ submodule with M/N' finite dimensional. The aim is to show that there exists N, a $\mathbf{C}K$ submodule with $\alpha(N) = 0$, and M/N finite dimensional.

We write $K'KK' = \sqcup k_i K' = \sqcup K'k_i$. Since K and K' are compact open, this is a finite union. The argument now is the same as that which established in Lemma 5.5. Define the maps

$$ k_i : N' \to M \to \frac{M}{N'}, \qquad k_i n' = k_n' + N' \quad \text{for } n' \in N', \tag{72}$$

and set

$$ N = \cap \ker k_i. \tag{73}$$

The arguments that: (i) N is a CK module, (ii) N is contained in $\ker \alpha$, and (iii) M/N is finite dimensional follow the pattern of Lemma 5.4. □

5.3. Application to loop algebras [1]

Let L be a finite dimensional simple Lie algebra over $\mathbf{C}$ and let $L[x, x^{-1}]$ denote the loop algebra of L consisting of Laurent polynomials in x with coefficients in L. A representation M of L is a projective representation with cocycle c if

$$(\xi x^i)(\psi x^j)m = ((\psi x^j)(\xi x^i)) + [\xi, \psi]x^{i+j}m + c(\xi x^i, \psi x^j)m \tag{74}$$

for all m in M. The representation is said to be of level k if it is projective with cocycle c given by

$$c(\xi x^i, \psi x^j) = ik\kappa(\xi, \psi)\delta_{i,-j}, \tag{75}$$

where $\kappa(,)$ is the Killing form on L.

A projective representation of $L[x, x^{-1}]$ corresponds to an ordinary representation of the central extension $L[x, x^{-1}] \oplus \mathbf{C}c$ in the usual way. Thus level κ representations are representations in which c acts as a multiplication by κ. In addition, there is an outer derivation d of $L[x, x^{-1}]$ given by

$$\mathrm{d}\xi x^i = i\xi x^i. \tag{76}$$

Form the Lie algebra $L[x, x^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d$, setting $[d, \xi x^i] = i\xi x^i, [d, c] = 0$. The algebras of interest are universal enveloping algebras of this Lie algebra and certain subalgebras. We write

$$\begin{aligned} U &= U(L[x, x^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d),\\ U_{\geq} &= U(L[x] \oplus \mathbf{C}c \oplus \mathbf{C}d),\\ U_{\leq} &= U(L[x^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d),\\ U_{>} &= U(L[x]x),\\ U_{<} &= U(L[x^{-1}]x^{-1}). \end{aligned} \tag{77}$$

The isomorphisms as vector spaces

$$L[x, x^{-1}] = L[x^{-1}]x^{-1} \oplus L[x] = L[x^{-1}]x^{-1} \oplus L \oplus L[x]x \tag{78}$$

induce isomorphisms of vector spaces

$$U = U_{<} \otimes U_{\geq} = U_{<} \otimes U(L \oplus \mathbf{C}c \oplus \mathbf{C}d) \otimes U_{>}. \tag{79}$$

The bracket with d provides a $\mathbf{Z}$ grading (as vector spaces) of all the universal enveloping algebras described here. With respect to this grading,

$$U_{<} = \oplus_{n \leq 0}(U_{<})_n, \tag{80}$$

where $(U_{<})_n$ is the set of elements of degree n. Each $(U_{<})_n$ is finite dimensional. Hence the subspace

$$(U_{<})_{(n)} = \oplus_{0 \leq j \leq n}(U_{<})_j \tag{81}$$

[1] See [5] for basic information on the subject.

is also finite dimensional and

$$U_< = \oplus_{n\leq 0}(U_<)_{(n)}. \tag{82}$$

Lemma 5.9. *$U_\geq$ is quasi-normal in U.*

Proof. This is essentially a consequence of the analogue of the Poincare–Birkhoff–Witt theorem for universal enveloping algebras. Observe that

$$(U_<)_{(n)} \otimes U_\geq = U_\geq \otimes (U_<)_{(n)}, \qquad U_\geq \otimes (U_<)_{(n)} = (U_<)_{(n)} \otimes U_\geq. \tag{83}$$

The result follows since a in U is in some $(U_<)_{(n)} \otimes U_\geq$. If $\{a_i\}$ is a basis for $(U_<)_{(n)}$, then

$$U_\geq a U_\geq = \sum_i a_i U_\geq = \sum_i U_\geq a_i \tag{84}$$

as required.

The anti-automorphism commonly used is that determined by the Lie algebra anti-automorphism $s: L[x, x^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d \to L[x, x^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d$,

$$s(\xi x^i) = -\xi x^{-i}, \qquad s(c) = c, \qquad s(d) = -d. \qquad \square \tag{85}$$

Proposition 5.10. *If M is a U module, then*
1. *$({}^{U_\geq}M)^*$ is a level k representation if M is,*
2. *$({}^{U_\geq}M)^*$ is locally finite as a $U_\geq$ module.*

Proof. (1) This is more or less a direct corollary of Lemma 5.9. Calculate

$$\begin{aligned}
&[(\xi x^i)(\psi x^j)\alpha - (\psi x^j)(\xi x^i)\alpha - [x, \psi]x^i + j\alpha](m) \\
&\quad = \alpha[(s(\psi x^j)(s(\xi x^i) - s(\xi x^i)s(\psi x^j) - s([x, \psi]x^{i+j}))m] \\
&\quad = \alpha[((\psi x^{-j})(\xi x^{-i}) - (\xi x^{-i})(\psi x^{-j}) - ([\psi, x]x^{-i-j}))m] \\
&\quad = \alpha[c(\psi x^{-j}, \xi x^{-i})m] = c(s(\psi x^j), s(\xi x^i))\alpha(m) \\
&\quad = -c(s(\xi x^i), s(\psi x^j))\alpha(m) = c(\xi x^i, \psi x^j)\alpha(m),
\end{aligned} \tag{86}$$

since $c(\xi x^i, \psi x^j) = ik\delta_{i,-j}\kappa(x, \psi) = -jk\delta_{-j,i}\kappa(\psi, x) = c(\psi x^{-j}, \xi x^{-i})$. This establishes (1).

For (2) observe that $s(U_\geq) = U_\geq$. The result then follows from Corollary 5.6. $\square$

Definition 4.11. Say a representation M of U is positive energy if d acts diagonally with real eigenvalues and the eigenvalues of d are bounded above.

As with the category of smooth representations of totally disconnected groups, so the category of positive energy representations of a loop algebra admits the existence of a dual. As the smooth dual of a representation of a totally disconnected group can be identified in terms of restricted comodules, so the dual positive energy representation of a representation M can be identified as an appropriate restriction of $({}^{U_\geq}M)^*$. It remains to identify the appropriate subcoalgebra of $(U_\geq)^*$.

The universal enveloping algebra $U_>$ has an augmentation ideal $U_>^+ = \oplus_{n>1}(U_>)_n$. This generates an ideal U_0 of $U_\geq$:

$$U_0 = U_\geq(U_>^+). \tag{87}$$

A short calculation shows that U_0 is in fact a two-sided ideal. Define

$$P_0^N = im\left(\frac{U_\geq}{(U_0)^N}\right)^* \to (U_\geq)^* \tag{88}$$

Since $P_0^{N+j} \geq P_0^N$, define

$$P_0 = \cup P_0^N. \tag{89}$$

Proposition 5.12.

1. ${}_{P_0}|({}^{U_\geq}M)^*$ *is a* U *submodule of* $({}^{U_\geq}M)^*$.
2. *If* ${}_{P_0}|({}^{U_\geq}M)^*$ *is generated by a finite set of eigenvectors for* d, *then it is positive energy*.

Proof. (1) Check that for z in U, q in ${}_{P_0}|({}^{U_\geq}M)^*$, $z \bullet q$ is in $P_0|({}^{U_\geq}M)^*$, or equivalently, for some N, any u in $s(U_0^N)$, $u \bullet z \bullet q = 0$. Using $P_0 = \cup P_0^N$, it can be shown that

$${}_{P_0}|({}^{U_\geq}M)^* = {}_{\cup P_0^N}|({}^{U_\geq}M)^*. \tag{90}$$

Suppose, q is in $P_0^{N'}|({}^{U_\geq}M)^*$ for some N', that is, $u \bullet q = 0$ for all u in $s(U_0^{N'})$. If $i \geq 0$, and z in $(U_\geq)i$, then $u \bullet z \bullet q = 0$, since z is in $U_\geq$ and $s(U_0^{N'})$ is an ideal of $U_\geq$. If $i < 0$, observe that

$$s(U_0^N)(U_\geq)_i \leq (U_\geq)_{(i)}s(U_0^{N+i}). \tag{91}$$

Thus for z in $(U_\geq)_i$, q in $P_0^{N'}|({}^{U_\geq}M)^*$, $u \bullet z \bullet q = 0$ provided $N > N' - i$.

(2) We write $V = {}_{P_0}|({}^{U_\geq}M)^*$. Assume that $\{q_i\}$ is a finite generating set for V of d eigenvectors. Since V is locally finite as $U_\geq$ module, $U_\geq\{q_i\}$ is a finite dimensional $U_\geq$ module, we call it D. In particular, the element d acts on D, and is diagonalizable on D with finitely many eigenvalues. But then since

$$V = UD = U_< D, \tag{92}$$

d acts diagonally on V and the eigenvalues are bounded below. □

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